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Trigonometry and Fibonacci numbers

BARRY LEWIS

This article sets out to explore some of the connections between two seemingly distinct mathematical objects: trigonometric functions and the integer sequences composed of the Fibonacci and Lucas numbers. It establishes that elements of Fibonacci/Lucas sequences obey identities that are closely related to traditional trigonometric identities. It then exploits this relationship by converting existing trigonometric results into corresponding Fibonacci/Lucas results. Along the way it uses mathematical tools that are not usually associated with either of these objects.

We start with a derivation of explicit formulas for each of these different objects.

De Moivre's formula

The addition formulae for cosine and sine give:

$$\cos r\theta = \cos(r-1)\theta\cos\theta - \sin(r-1)\theta\sin\theta$$

$$\sin r\theta = \sin(r-1)\theta\cos\theta + \cos(r-1)\theta\sin\theta$$

and this may be written in the matrix form

$$\begin{pmatrix}\cos r\theta\\ \sin r\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos(r-1)\theta\\ \sin(r-1)\theta\end{pmatrix};$$

repeated use then gives

$$\begin{pmatrix}\cos r\theta\\ \sin r\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}^{r-1}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}. \tag{1}$$

Now we use the methods of linear algebra to find a way of evaluating the powers of this matrix:

(i) the eigenvalues λ are given by

$$\begin{vmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

and this equation has the intriguing complex roots

$$\lambda = \cos\theta \pm i\sin\theta.$$

(ii) the eigenvectors are then given by

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \left(\cos\theta \pm i\sin\theta\right)\begin{pmatrix}x\\ y\end{pmatrix}$$

and this reduces to

$$y = \pm ix.$$

So the required eigenvectors are

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

(iii) we easily find that

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix}.$$

(iv) this leads to the matrix identity

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix}.$$

Plugging this into (1) now gives

$$\begin{pmatrix} \cos r\theta \\ \sin r\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{r-1} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix} \right)^{r-1} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} (\cos\theta + i\sin\theta)^{r-1} & 0 \\ 0 & (\cos\theta - i\sin\theta)^{r-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

and, after multiplying out the right-hand side and some simplification, we find that:

$$\sin r\theta = \tfrac{1}{2i}\left((\cos\theta + i\sin\theta)^r - (\cos\theta - i\sin\theta)^r\right)$$

$$\cos r\theta = \tfrac{1}{2}\left((\cos\theta + i\sin\theta)^r + (\cos\theta - i\sin\theta)^r\right).$$

It is a simple consequence of this that

$$\cos r\theta + i\sin r\theta = (\cos\theta + i\sin\theta)^r$$

and De Moivre's formula miraculously emerges. In passing, we recall a fundamental property of the trigonometric functions

$$\cos^2\theta + \sin^2\theta = 1.$$

Binet's formula

The Fibonacci sequence needs no introduction, but the sibling Lucas sequence may not be so familiar. These sequences obey the same recurrence relation – each term is the sum of the two previous terms – but have different initial terms

$$\{F_r\} = \{0, 1, 1, 2, 3, 5, 8, \dots\} \text{ and } \{L_r\} = \{2, 1, 3, 4, 7, 11, 18, \dots\}.$$

Using the recurrence relations

$$F_r = F_{r-1} + F_{r-2} \qquad \text{and} \qquad L_r = L_{r-1} + L_{r-2}$$

and induction, it is an easy task to prove that

$$2F_r = F_{r-1} + L_{r-1} \qquad \text{and} \qquad 2L_r = 5F_{r-1} + L_{r-1}.$$

We may write this in the matrix form

$$\begin{pmatrix} L_r \\ F_r \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} L_{r-1} \\ F_{r-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} L_{r-1} \\ F_{r-1} \end{pmatrix},$$

and repeated use then gives

$$\begin{pmatrix} L_r \\ F_r \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^{r-1}\begin{pmatrix} L_1 \\ F_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

We use the same methods of linear algebra:

(i) the eigenvalues are given by

$$\begin{vmatrix} \frac{1}{2} - \lambda & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0.$$

Let us write the roots of this equation in the form

$$\phi_1 = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \phi_2 = \frac{1 - \sqrt{5}}{2}.$$

(ii) the eigenvectors are given by

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \phi_i\begin{pmatrix} x \\ y \end{pmatrix}, \qquad i = 1,\ 2,$$

which reduce to

$$y = \pm\frac{1}{\sqrt{5}}x,$$

so that the eigenvectors are

$$\begin{pmatrix} 1 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{5}} \end{pmatrix}.$$

(iii) we find that

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{2} \end{pmatrix}.$$

(iv) we may write

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{2} \end{pmatrix}$$

so that

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^{r-1} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \phi_1^{r-1} & 0 \\ 0 & \phi_2^{r-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{2} \end{pmatrix}.$$

Using this in (2) we now have

$$\begin{pmatrix} L_r \\ F_r \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \phi_1^{r-1} & 0 \\ 0 & \phi_2^{r-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}}{2} \\ \frac{1}{2} & -\frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which on expansion leads to Binet's explicit formula for the Fibonacci numbers and the corresponding formula for the Lucas numbers:

$$F_r = \frac{1}{\sqrt{5}}\left(\phi_1^r - \phi_2^r\right) \qquad \text{and} \qquad L_r = \frac{1}{\sqrt{5}}\left(\phi_1^r + \phi_2^r\right).$$

But there is a hidden treasure in this. As a simple consequence of these explicit forms, we deduce that:

$$\frac{L_r + \sqrt{5}F_r}{2} = \phi_1^r \Rightarrow \left(\frac{L_r + \sqrt{5}F_r}{2}\right)^n = \phi_1^{nr} = \frac{L_{nr} + \sqrt{5}F_{nr}}{2};$$

similarly

$$\frac{L_r - \sqrt{5}F_r}{2} = \phi_2^r \Rightarrow \left(\frac{L_r - \sqrt{5}F_r}{2}\right)^n = \frac{L_{nr} - \sqrt{5}F_{nr}}{2};$$

which were first published in [1]. Finally, we note that

$$L_r^2 - 5F_r^2 = 4\phi_1^r\phi_2^r = 4(\phi_1\phi_2)^r = 4(-1)^r.$$

A geometric interpretation

The similarity of these results has a revealing interpretation. The two basic trigonometric functions are simply the coordinate functions of a point on the unit circle.

The matrix we have iterated

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has a simple geometric meaning: it maps the unit circle into itself – it is a rotation of θ radians.

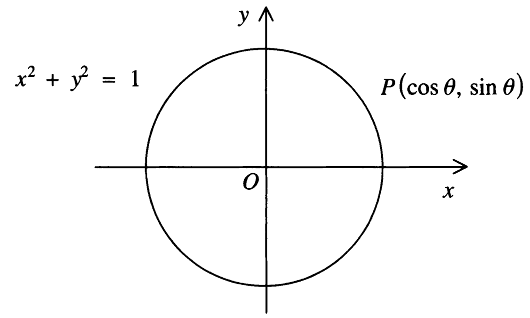

FIGURE 1

The Fibonacci and Lucas numbers have a corresponding geometric role. When r is even, the terms of the sequences are the coordinate functions of integer points on the positive portion of one hyperbola, and when r is odd they are the co-ordinate functions of integer points on the positive portion of another hyperbola; these hyperbola share a common asymptote.

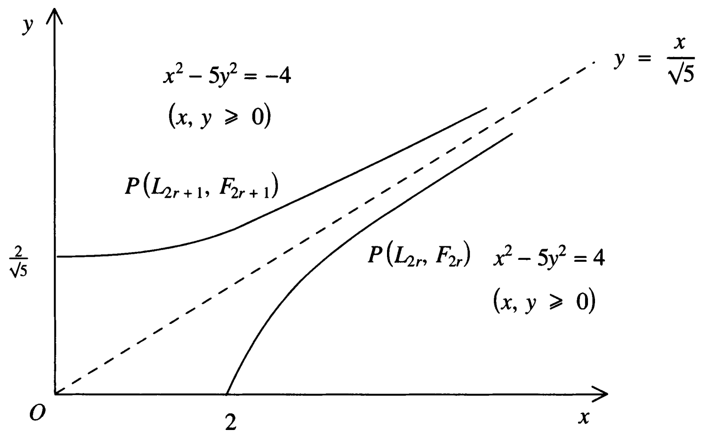

FIGURE 2

This time the corresponding matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

maps one leg of each hyperbola into the other.

Fundamental identities

First we look at the traditional formulas of trigonometry and use these to find corresponding Fibonacci/Lucas counterparts. We start with the addition formula for sine:

$$\sin(z + w) = \sin z \cos w + \sin w \cos z.$$

We are led to examine:

$$F_rL_s + F_sL_r$$
$$= \tfrac{1}{\sqrt{5}}\left(\left(\phi_1^r - \phi_2^r\right)\left(\phi_1^s + \phi_2^s\right) + \left(\phi_1^s - \phi_2^s\right)\left(\phi_1^r + \phi_2^r\right)\right)$$
$$= \tfrac{1}{\sqrt{5}}\left(\phi_1^{r+s} + (-1)^s\phi_1^{r-s} - (-1)^s\phi_2^{r-s} - \phi_2^{r+s} + \phi_1^{r+s} + (-1)^s\phi_1^{r-s} - (-1)\phi_2^{r-s} - \phi_2^{r+s}\right)$$
$$= 2F_{r+s}.$$

This becomes

$$F_{r+s} = \tfrac{1}{2}\left(F_rL_s + F_sL_r\right).$$

What about the cosine of a sum? We have

$$\cos(z + w) = \cos z \cos w - \sin z \sin w.$$

We might expect that this should come from

$$L_rL_s - F_rF_s$$
$$= \left(\phi_1^r + \phi_2^r\right)\left(\phi_1^s + \phi_2^s\right) - \frac{1}{5}\left(\phi_1^r - \phi_2^r\right)\left(\phi_1^s - \phi_2^s\right)$$
$$= \phi_1^{r+s} + (-1)^s\phi_1^{r-s} + (-1)^s\phi_2^{r-s} + \phi_2^{r+s} - \tfrac{1}{5}\left(\phi_1^{r+s} - (-1)^s\phi_1^{r-s} - (-1)\phi_2^{r-s} + \phi_2^{r+s}\right)$$

but we need to modify it

$$L_rL_s - 5F_rF_s$$
$$= \left(\phi_1^r + \phi_2^r\right)\left(\phi_1^s + \phi_2^s\right) - \frac{1}{5}\left(\phi_1^r - \phi_2^r\right)\left(\phi_1^s - \phi_2^s\right)$$
$$= \phi_1^{r+s} + (-1)^s\phi_1^{r-s} + (-1)^s\phi_2^{r-s} + \phi_2^{r+s} - \phi_1^{r+s} + (-1)^s\phi_1^{r-s} + (-1)\phi_2^{r-s} - \phi_2^{r+s}$$
$$= 2(-1)^s L_{r-s}.$$

So

$$L_{r-s} = \frac{(-1)^s}{2}\left(L_rL_s - 5F_rF_s\right)$$

in which it is assumed that $s \leqslant r$.

Here is a complete list of Fibonacci/Lucas identities, looking for all the world like identities from a trigonometric textbook. Although many of these formulas have been published before (for example [2]) they don't appear to have been presented in a manner that emphasises their 'trigonometric' character. Throughout, it is assumed that $s \leqslant r$; in the sum/difference formulas we also require that r and s should have the same parity.

Binet's formulas

$$L_r = \phi_1^r + \phi_2^r$$
$$\sqrt{5}F_r = \phi_1^r - \phi_2^r$$

Fundamental identity

$$L_r^2 - 5F_r^2 = 4(-1)^r$$

Addition formulas

$$2F_{r+s} = F_rL_s + F_sL_r$$
$$2(-1)^s F_{r-s} = F_rL_s - F_sL_r$$
$$2L_{r+s} = L_rL_s + 5F_rF_s$$
$$2(-1)^s L_{r+s} = L_rL_s - 5F_rF_s$$

Sum/difference

$$F_r + (-1)^{\frac{1}{2}(r-s)}F_s = F_{\frac{1}{2}(r+s)}L_{\frac{1}{2}(r-s)}$$
$$F_r - (-1)^{\frac{1}{2}(r-s)}F_s = F_{\frac{1}{2}(r-s)}L_{\frac{1}{2}(r+s)}$$
$$L_r + (-1)^{\frac{1}{2}(r-s)}L_s = L_{\frac{1}{2}(r+s)}L_{\frac{1}{2}(r-s)}$$
$$L_r - (-1)^{\frac{1}{2}(r-s)}L_s = 5F_{\frac{1}{2}(r+s)}F_{\frac{1}{2}(r-s)}$$

Products

$$5F_rF_s = L_{r+s} - (-1)^sL_{r-s}$$
$$F_rL_s = F_{r+s} + (-1)^sF_{r-s}$$
$$L_rL_s = L_{r+s} + (-1)^sL_{r-s}$$

'De Moivre' formula

$$\left(\frac{L_r \pm \sqrt{5}F_r}{2}\right)^n = \frac{L_{nr} \pm \sqrt{5}F_{nr}}{2}.$$

Applications

There are many examples in which a classical result in trigonometry can inspire a Fibonacci counterpart. This first one provides a very surprising result. It is inspired by the trigonometric identity

$$\sin\theta = 2^n \sin\tfrac{1}{2^n}\theta \cos\tfrac{1}{2^n}\theta \cos\tfrac{1}{2^{n-1}}\theta \ldots \cos\tfrac{1}{2}\theta.$$

This is a simple consequence of the double angle formula for the sine

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta,$$
$$= 2^2\sin\tfrac{1}{4}\theta\cos\tfrac{1}{4}\theta\cos\tfrac{1}{2}\theta;$$

and then, inductively, we obtain

$$\sin\theta = 2^n \sin\tfrac{1}{2^n}\theta \cos\tfrac{1}{2^n}\theta \cos\tfrac{1}{2^{n-1}}\theta \ldots \cos\tfrac{1}{2}\theta.$$

as required. The Fibonacci counterpart of this comes from the first addition formula

$$2F_{2n} = 2F_{n+n} = F_nL_n + L_nF_n$$

so that

$$F_{2n} = F_nL_n.$$

Then we have

$$F_{2^rn} = F_{2^{r-1}n}L_{2^{r-1}n} = F_{2^{r-2}n}L_{2^{r-2}n}L_{2^{r-1}n}$$

and, inductively, we obtain the beautiful factorisation

$$F_{2^rn} = F_n\prod_{k=1}^{r} L_{2^{r-k}n},$$

an unexpected gem.

Another example is based on a 'telescoping' sum

$$\sum_{k=0}^{r} \frac{1}{\cos k\theta \cos (k+1)\theta} = \frac{\tan (r+1)\theta}{\sin \theta}.$$

This telescopes because

$$\begin{aligned} &\cos r\theta \cos (r+1)\theta \big(\tan (r+1)\theta - \tan r\theta\big) \\ &= \sin (r+1)\theta \cos r\theta - \cos (r+1)\theta \sin r\theta \\ &= \sin\big((r+1)\theta - r\theta\big) = \sin\theta \end{aligned}$$

and hence

$$\begin{aligned} \sum_{k=0}^{r} \frac{\sin\theta}{\cos k\theta \cos (k+1)\theta} &= \big(\tan\theta - \tan 0\big) + \ \ldots \ + \big(\tan (r+1)\theta - \tan r\theta\big) \\ &= \tan (r+1)\theta. \end{aligned}$$

as required.

In the Fibonacci counterpart, we start with the difference between successive Fibonacci 'tangents'

$$\frac{F_{(r+1)n}}{L_{(r+1)n}} - \frac{F_{rn}}{L_{rn}} = \frac{F_{(r+1)n}L_{rn} - L_{(r+1)n}F_{rn}}{L_{(r+1)n}L_{rn}}.$$

The numerator of the latter expression may be simplified by using one of the addition identities

$$F_{(r+1)n}L_{rn} - L_{(r+1)n}F_{rn} = 2(-1)^{rn} F_n.$$

So we may write

$$\frac{F_{(r+1)n}}{L_{(r+1)n}} - \frac{F_{rn}}{L_{rn}} = \frac{2(-1)^{rn} F_n}{L_{(r+1)n}L_{rn}}$$

and then the sum 'telescopes' to give

$$\sum_{k=0}^{r} \frac{(-1)^{kn}}{L_{(k+1)n}L_{kn}} = \frac{F_{(r+1)n}}{2F_n L_{(r+1)n}}$$

as required.

This sum has an infinite form (courtesy of the Ratio Test). We have

$$\begin{aligned} \lim_{r\to\infty} \frac{F_{(r+1)n}}{L_{(r+1)n}} &= \lim_{r\to\infty} \frac{\phi_1^{(r+1)n} - \phi_2^{(r+1)n}}{\sqrt{5}\left(\phi_1^{(r+1)n} + \phi_2^{(r+1)n}\right)} = \frac{1}{\sqrt{5}} \lim_{r\to\infty} \frac{1 - \left(\left(\frac{\phi_2}{\phi_1}\right)^n\right)^{r+1}}{1 + \left(\left(\frac{\phi_2}{\phi_1}\right)^n\right)^{r+1}} \\ &= \frac{1}{\sqrt{5}} \text{ since} \left|\left(\frac{\phi_2}{\phi_1}\right)^n\right| < 1. \end{aligned}$$

So we have

$$\sum_{k=0}^{\infty} \frac{(-1)^{kn}}{L_{(k+1)n}L_{kn}} = \frac{1}{2F_n\sqrt{5}}.$$

In particular when $n = 1, 2$ respectively, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^{k}}{L_{k+1}L_{k}} = \frac{1}{2\sqrt{5}} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{1}{L_{2k+2}L_{2k}} = \frac{1}{2\sqrt{5}}.$$

That Fibonacci/Lucas sequences have the 'De Moivre' property suggests that trigonometric results that also rely on it will have a Fibonacci counterpart. To illustrate the process, I have drawn inspiration from a traditional textbook. The following question comes from [3] and calls for a closed form for the sum

$$C = \sum_{k \geqslant 0} \binom{r}{k} \cos 2k\theta.$$

There is a standard way of evaluating such sums based on De Moivre's formula. First we create two parts, one of which is *real* and one *imaginary*:

$$C = \sum_{k \geqslant 0} \binom{r}{k} \cos 2k\theta \qquad \text{and} \qquad iS = \sum_{k \geqslant 0} \binom{r}{k} i \sin 2k\theta.$$

Then we work with their sum

$$\begin{aligned} C + iS &= \sum_{k \geqslant 0} \binom{r}{k} \cos 2k\theta + \sum_{k \geqslant 0} \binom{r}{k} i \sin 2k\theta = \sum_{k \geqslant 0} \binom{r}{k} (\cos\theta + i\sin\theta)^{2k} \\ &= \left(1 + (\cos\theta + i\sin\theta)^2\right)^r \\ &= (\cos\theta + i\sin\theta)^r \left((\cos\theta + i\sin\theta)^{-1} + (\cos\theta + i\sin\theta)\right)^r \\ &= \left(\cos r\theta + i\sin r\theta\right)\left((\cos\theta - i\sin\theta) + (\cos\theta + i\sin\theta)\right)^r \\ &= \left(\cos r\theta + i\sin r\theta\right) 2^r \cos^r\theta. \end{aligned}$$

Now, extracting the real and imaginary parts gives

$$\sum_{k \geqslant 0} \binom{r}{k} \cos 2k\theta = 2^r \cos^r\theta \cos r\theta \text{ and } \sum_{k \geqslant 0} \binom{r}{k} \sin 2k\theta = 2^r \cos^r\theta \sin r\theta. \quad (3)$$

The Fibonacci/Lucas counterpart is derived by identical means – but it has a new term. First we define the *Golden* (irrational) and the *Base* (rational) parts

$$\sqrt{5}F = \sum_{k \geqslant 0} \binom{r}{k} (-1)^{(r+k)n} \sqrt{5} F_{2kn} \text{ and } L = \sum_{k \geqslant 0} \binom{r}{k} (-1)^{(r+k)n} L_{2kn}.$$

Again we evaluate the sum

$$L+\sqrt{5}F = \sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}\left(L_{2kn}+\sqrt{5}F_{2kn}\right) = \sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}\left(2\phi_1^{2kn}\right)$$

$$= 2(-1)^{rn}\sum_{k\geqslant 0}\binom{r}{k}\left((-1)^n\phi_1^{2n}\right)^k = 2(-1)^{rn}\left(1+(-1)^n\phi_1^{2n}\right)^r$$

$$= 2(-1)^{rn}\phi_1^{rn}\left(\phi_1^{-n}+(-1)^n\phi_1^n\right)^r = 2(-1)^{rn}\phi_1^{rn}\left(\left(\frac{-1}{\phi_2}\right)^{-n}+(-1)^n\phi_1^n\right)^r$$

$$= 2\phi_1^{rn}\left(\phi_2^n+\phi_1^n\right)^r = \left(L_{rn}+\sqrt{5}F_{rn}\right)L_n^r.$$

Now we 'extract' the Base and Golden parts so that

$$F = \sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}F_{2kn} = L_n^rF_{nr} \text{ and } L = \sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}L_{2kn} = L_n^rL_{nr}.$$

This leads to the symmetric identity

$$L_{nr}\sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}F_{2kn} \;=\; F_{nr}\sum_{k\geqslant 0}\binom{r}{k}(-1)^{(r+k)n}L_{2kn}.$$

We end this section, as we began the article, with matrices. The first equation of (3) may be written in the form

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ . & . & . & . & . \end{pmatrix}\begin{pmatrix} 1 \\ \cos 2\theta \\ \cos 4\theta \\ \cos 6\theta \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ 2\cos\theta\cos\theta \\ 2^2\cos^2\theta\cos 2\theta \\ 2^3\cos^3\theta\cos 3\theta \\ . \end{pmatrix}.$$

The inverse of the *Pascal* matrix is particularly simple, and leads to the dual result

$$\begin{pmatrix} 1 \\ \cos 2\theta \\ \cos 4\theta \\ \cos 6\theta \\ . \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ . & . & . & . & . \end{pmatrix}\begin{pmatrix} 1 \\ 2\cos\theta\cos\theta \\ 2^2\cos^2\theta\cos 2\theta \\ 2^3\cos^3\theta\cos 3\theta \\ . \end{pmatrix}$$

which then gives

$$\cos 2r\theta \;=\; \sum_{k\geqslant 0}(-1)^{r+k}\binom{r}{k}2^k\cos^k\theta\cos k\theta.$$

There is also a sine version of this:

$$\sin 2r\theta \;=\; \sum_{k\geqslant 0}(-1)^{r+k}\binom{r}{k}2^k\cos^k\theta\sin k\theta.$$

The Fibonacci/Lucas counterpart is

$$F_{2nr} = \sum_{k \geqslant 0} \binom{r}{k} (-1)^{(r+k)(n-1)} L_n^k F_{nk} \text{ and } L_{2nr} = \sum_{k \geqslant 0} \binom{r}{k} (-1)^{(r+k)(n-1)} L_n^k L_{nk}.$$

Conclusion

It is worth manipulating Binet's formula to see why Fibonacci/Lucas numbers exhibit this unexpected propensity to trigonometric behaviour. We have

$$L_r = \phi_1^r + \phi_2^r = \frac{2i^r}{2(\phi_1\phi_2)^{\frac{1}{2}r}} (\phi_1^r + \phi_2^r) = 2i^r \frac{1}{2} \left(\frac{\phi_1^r + \phi_2^r}{(\phi_1\phi_2)^{\frac{1}{2}r}} \right)$$

$$= 2i^r \frac{1}{2} \left(\left(\frac{\phi_1}{\phi_2} \right)^{\frac{1}{2}r} + \left(\frac{\phi_1}{\phi_2} \right)^{-\frac{1}{2}r} \right) = 2i^r \frac{1}{2} \left(\exp\left(\frac{r}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right) + \exp\left(-\frac{r}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right) \right)$$

$$= 2i^r \cosh\left(\frac{r}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right).$$

But we have

$$\cosh z = \cos iz$$

and so

$$L_r = 2i^r \cos\left(\frac{ir}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right).$$

The corresponding result for Fibonacci numbers is

$$F_r = \frac{2i^r}{\sqrt{5}} \sinh\left(\frac{r}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right) = -\frac{2i^{r+1}}{\sqrt{5}} \sin\left(\frac{ir}{2} \log\left(\frac{\phi_1}{\phi_2} \right) \right).$$

This explains both the trigonometric and the hyperbolic aspects of the behaviour of Fibonacci/Lucas numbers.

There remains one conundrum. The appearance of the Fibonacci sequence in nature is well known and extensive. But where is the Lucas sequence? Surely, nature doesn't prefer one to the other – or does it?

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References

1. S. Fisk, *The Fibonacci Quarterly* **1** (April 1963) p. 85.
2. L. Taylor, *The Fibonacci Quarterly* **20** (November 1982) p. 369.
3. M. Gow, *A course in pure mathematics*, Edward Arnold (1992), p. 166, Question 28 (i).

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